group of order 8 which can be represented on four letters. The latter may again be distinguished by the fact that only one of them contains operators of order 12. This concludes the examination of all the possible cases and proves that the total number of groups of order 168 that have 21 subgroups of order 8 is the same as the number of those containing 7 such subgroups, viz. 16. *The total number of groups of order* 168 *is therefore* 57. Only one of these is insolvable.

ON LIMITS.

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In most of the elementary text-books of algebra and calculus the chapters on limits are treated in a rather indefinite manner. I suspect that the reason for this deficiency lies partly in the semi-philosophical nature of the subject, partly in the neglect of the authors to apply the results of the theory of functions to limitting processes.

As an example I mention the frequently occurring definition of the *limit of a variable.*

(*a*) "When according to its law of change, a variable approaches indefinitely near a constant, but can never reach it, the constant is called the limit of the variable."

This definition restricts the variable to the members of a sequence as it will appear from the following proposition given by Harkness and Morley :*

I. "The numbers ξ_1, ξ_2, ξ_3, ... of a sequence are said to tend to the limit a, when to every positive number ε there corresponds a positive integer μ such that for ξ_μ and for all later members ξ_n of the sequence we have

$$| \xi_n - a | < \varepsilon."$$

a itself does not belong to the sequence. Terms in the sequence of all positivs integers 1, 2, 3, ...—, ∞ does not belong to the sequence, although ∞ is the limit of the sequence. Similarly, in the sequence

$$\tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \ldots.$$

0, which is its limit, does not belong to the sequence. Notice also that 0 and ∞, which shall be defined presently, are among the values which a limit can have.

An infinitely large quantity, or the symbol ∞ is defined as the indefinite quantity A which satisfies the inequality

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**Introduction to Analytic Functions*, page 67.
See also Burkhardt: *Functionentheorie*, Vol. I., page 68.

$$A > N$$

for all positive numbers N no matter how large. Here, N may be either an integer or any other positive real number. $-\infty$ may be defined in a similar manner.

For a sharp conception of infinite limits as used in the theory of functions and in geometry, a distinction between *absolute infinity* and *variable infinity* (Werdendes Unennlich) should be made.*

Designating the first infinity by ∞, the second by (∞) we have the definitions

$$\frac{1}{\infty}=0 \text{ (absolute zero),}$$

$$\frac{1}{(\infty)}=\varepsilon \text{ (infinitesimal).}$$

An infinitesimal ε itself is defined by the inequality

$$|\varepsilon| < |a|,$$

for all values of a no matter how small. Conversely

$$\frac{1}{\varepsilon}=(\infty); \ \frac{1}{0}=\infty.\dagger$$

Take now a variable y whose values depend upon an independent variable x of which we assume that it may assume all real values of the interval between a and b. This interval may be closed, *i e.*, a and b belong to the interval; it may be open, in which case a and b are excluded; it may be closed at a and open at b, or conversely. In signs we have the four cases

$$a \leqq x \leqq b,$$
$$a<x<b,$$
$$a \leqq x<b,$$
$$a<x \leqq b.$$

Among the values which a and b may have we also include the symbols 0, $+\infty$, $-\infty$, and as a matter of course ε, $+(\infty)$, $-(\infty)$. From the physical standpoint 0, $+\infty$, $-\infty$ are pure abstractions of the human mind, like the hypotheses of the infinite divisibility of space. A real variable x which successively may take all values of an interval is said to be continuous. For the subsequent applications I add the following propositions:

**Encyklopaedie der Mathematischen Wissenschaften*, Vol. I., Heft 1., IA3., page 68.
†Burkhardt, loc. cit., pages 27—29.

II. A function $f(x)$ is called *continuous* at a certain place x_0, if the difference

$$f(x_0+h)-f(x_0),$$

which is a function of h, becomes infinitely small with h. (If x is not variable unrestrictedly, the variability of h has to be restricted to such values for which x_0+h belongs to the values of x.

III. A function is called *continuous within an interval* if it is continuous at every place of the interval.

IV. A function which is continuous within an interval reaches its upper and lower limits within this interval.*

Thus taking $y=1/x$, $0<x\leqq\infty$, (∞) is an indefinite upper limit, while 0 is the lower limit. For $-\infty\leqq x<0$, the upper and lower limits are 0 and $-(\infty)$ respectively. The point 0 is a point of discontinuity, for in this case $f(0+h)-f(0)$, as h approaches 0, may assume any value. However,

$$\lim_{h\doteq 0}\{f(\varepsilon+h)-f(\varepsilon)\}=0,$$

i. e. $y=1/x$ is continuous for infinitesimal values of x. The difference between this example and the sequence $\frac{1}{1}$, $\frac{1}{2}$. $\frac{1}{3}$, $\frac{1}{4}$, ... is now apparent. According as a variable is continuous or discontinuous within an interval it may or may not reach its limit, or a limit. It may even be alternately greater or less than its limit, without reaching it, as in the case of

$$\sum_0^n(-1)^n a_n \text{ (n integer, a_n's decreasing),}$$

or the convergents of a continued fraction.

Hilbert in a lecture on "Mathematical Problems" before the International Mathematical Congress at Paris, 1900, said :†

"In der Geometrie gelingt der Nachweis der Widerspruchslosigkeit der Axiome dadurch, dass man einen geeigneten Bereich von Zahlen konstruiert, derart, dass den geometrischen Axiomen analoge Beziehungen zwischen den Zahlen dieses Bereiches entsprechen und dass demnach jeder Widerspruch in den Folgerungen aus geometrischen Axiomen auch in der Arithmetik jenes Zahlenbereiches erkennbar sein müsste."

With this purpose in view I shall apply the theorems on limits to a few geometrical problems.

The previous statements concerning the limits of $y=1/x$ may also be expressed thus :

$$\lim_{x=\pm\varepsilon}.\left(\frac{1}{x}\right)=\pm(\infty);\quad \lim_{x=\pm\infty}.\left(\frac{1}{x}\right)=0.$$

***Burkhardt, loc. cit., page 71, and Harkness and Morley, page 74.**

†*Archiv der Mathematik und Physik*, III Series, Vol. I., page 55.

Geometrically $y=1/x$ represents an hyperbola having the coördinate-axes as asymptotes. The variable x can evidently assume all values of the closed interval $-\infty \leqq x \leqq +\infty$. As x approaches 0 indefinitely (infinitesimally, ε) from positive values, y reaches a positive infinite value, or becomes (∞), infinitely large. If x approaches 0 indefinitely from negative values, y reaches $-(\infty)$.

Now the behavior of y at $x=\pm\infty$, $(\pm(\infty))$, is exactly the same as that of x at $y=\pm\infty$, $(\pm(\infty))$. Thus, $y=0$ for $x=\pm\infty$ gives us the notion of projective geometry that the asymptotes of an hyperbola are the tangents at its infinitely distant points. Arithmetically, an hyperbola has four infinite points, ε, (∞), ε, $-(\infty)$, (∞), ε, $-(\infty)$, ε. This however does not agree with the conceptions of an asymptote as a tangent and it is therefore necessary to assume the axiom that *a straight line has only one infinite point* and that *a plane has only one infinite straight line. This axiom is equivalent with the axiom of absolute infinity.* Indeed, $y=\frac{1}{0}=\infty$ and $y=1/\infty=0$ now signifies that the hyperbola has only two infinite points and that the asymptotes are the tangents at these two points.

We have now perfect agreement with the results of perspective collineation. If a circle intersects the axis of collineation s at two points A and B, its perspective is an hyperbola whose asymptotes are the transformed tangents at A and B. Perspective is a *one to one* correspondence, so that from this point of view the asymptotes necessarily have to be considered as limitting positions of tangents. By an inversion, which realizes also a (1, 1) correspondence, the hyperbola $xy=1$ is transformed into the lemniscate

$$xy=(x^2+y^2)^2.$$

This curve is continuous at $x=y=0$; to the asymptotes correspond the tangents to the lemniscate at the origin. Thus, in the theory of functions and in geometry the treatment of functions at $x=y=0$, respectively $z=0$, is not essentially different from that at infinity. In geometry a collineation reduces a problem concerning infinities to one of finite regions. In the theory of functions this is done by the inversion $u=1/z$. In the theory of assemblages,* ∞ is a place of accumulation (Hänfungstelle) for the system 1, 2, 3, 4, ... If we take a *non-enumerable* system consisting of all real numbers, ∞ is not more of a place of accumulation than any other point. As a second example take a polygon of n sides inscribed to a circle of unit-diameter. For the perimeter of this polygon we have

$$S_n=n\sin\frac{\pi}{n}$$

For $n=2, 3, \ldots$ we have a sequence whose lower limit is 2 and whose upper limit is

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*See E. Borel: *Lecons sur la theorie de fonctions*, Vol. 1., pages 1—49.

$$\lim_{n \doteq \infty}(S_n) = (\infty.0) = \lim_{n \doteq \infty}. \left\{ \frac{-\frac{\pi}{n^2}}{-\frac{1}{n^2}} \right\} = \pi.$$

This limit does not belong to the sequence, *i. e.* none of the polygons is equal to the circumference. We may, however, say correctly, as Sophus Lie does,* that in the limit-process (Grenzübergang) the polygon goes over into the circle.

The matter stands quite different if we consider the function

$$y = x\sin\frac{\pi}{x}$$

of the real continuous variable x. The function is continuous within the interval

$$0 < x \leqq \infty,$$

and defined within the interval

$$0 \leqq x < \infty.$$

It is noticed that $x=0$ is a singular point for the function, although $y=0$ for $x=0$. We have

$$\frac{dy}{dx} = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}.$$

As x approaches 0 indefinitely, $\frac{dy}{dx}$ changes infinitely many times abruptly from $+(\infty)$ to $-(\infty)$. As x passes through zero, y reaches the limit 0, owing to the circumstance that x appears as a factor in the expression for y. Geometrically, we may say that the curve represented by $y=x\sin\frac{\pi}{x}$ has an infinite number of infinitely small oscillations infinitely close to the origin.

For $x=\infty$, y is not defined, but its value is $\lim_{x=\infty}.(y) = \lim_{x=\infty}.\left(x\sin\frac{\pi}{x}\right) = \pi$. This upper limit† of y is attained for $x=\infty$. The straight line $y=\pi$ is an asymptote of the given curve and may be considered as a tangent at its infinite point. The inverse curve (origin as center) is continuous $x=0$; discontinuous at $x=\infty$.

**Beruehrungstransformationen*, Vol. I., page —.

†Within the interval ($2 \leqq x \leqq \infty$).